Modelling long term interest rates for pension funds

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Joint work with Jan de Kort
Overview

- Provisions for funded pension system
- Inter- and extrapolation problems for long term discounting
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- Extrapolation using an ultimate forward rate assumption
- Alternative formulations
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- Extrapolation using an ultimate forward rate assumption
- Alternative formulations
- Conclusions & Future Research
Motivation

In collective funded pension schemes which provide annuities at retirement, the participants share

- **Interest rate Risk**, since price of funding long-term liabilities depends on current term structure in market-consistent approach.

- **Equity Risk**, when proceeds are partially invested in stocks in an attempt to compensate for inflation in pension payments.

- **Longevity Risk**, since expected remaining lifetime at pension age is currently increasing over time.
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- Buffers above what is needed for indexation of existing pensions are kept for younger generations (dampening of effects of overfunding)
- Longevity risk is currently unidirectional and highly correlated across ages so diversifying risk over generations seems less effective.
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- Being overly optimistic in valuation is beneficial for older participants (and for pension fund managers?)
- Being overly pessimistic in valuation is beneficial for younger participants (and for regulators?)
Motivation

Incorporating market information whenever possible is useful in the search for objective criteria.

- But bond prices and swap rates are not available beyond a certain maximal maturity.

Concrete subproblem in this talk: how can we use market prices for fixed income products to generate discount curves that extrapolate beyond maturities for which reliable information is available?
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- In times of severe market distress, even shorter maturities may not give consistent information.
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how can we use market prices for fixed income products to generate discount curves that extrapolate beyond maturities for which reliable information is available?
Approach of European Insurers

To generate official discount curves European insurance regulator EIOPA uses:
- information from coupon bonds or swap quotes for maturities up until 20 yrs
  - interpolation (up until maturity 20 yrs)
  - extrapolation (from 20 to 60 yrs)

The UFR (ultimate forward rate) is assumed to be constant although the evidence for this is limited.

We propose methods to estimate asymptotic forward rates which are consistent with the methodology proposed by EIOPA but without the assumption that the UFR is constant.

This allows us to check that assumption using unsmoothed market information of liquid tradeable assets without making additional assumptions on model structure.
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Denote the amount to be paid at a time $t \geq 0$ to receive a certain single euro at time $T \geq t$, the zero-coupon bond price, by $p(t, T)$. \[ p(t, T) = \exp(-\int_{t}^{T} f(t, u) \, du). \]
Preliminaries

- Denote the amount to be paid at a time $t \geq 0$ to receive a certain single euro at time $T \geq t$, the zero-coupon bond price, by $p(t, T)$.
- The continuous-time yield $y(t, T)$ and forward rate $f(t, T)$ are then implicitly defined by

$$p(t, T) = \exp(-(T-t)y(t, T)) = \exp(-\int_t^T f(t, u) du).$$
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Assume given fixed income instruments indexed by $i \in I$ which pay cashflows $c_{ij}$ at times $u_j$ ($j \in J$) and have a current price $m_i$. An interpolating curve $\bar{p}(0, t)$ must thus satisfy

$$m_i = \sum_{j \in J} c_{ij} \bar{p}(0, u_j).$$
Schweikert functions as solution to interpolation problem

- Interpolating curve is chosen by EIOPA to have the form

\[ \tilde{p}(0, t) = (1 + g(t))e^{-f_{\infty}t}, \quad g(t) = \sum_{j \in \mathcal{J}} \eta_j W(t, u_j) \]
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with \( f_\infty \) the UFR, \( (\eta_j)_{j \in J} \) appropriately chosen weights and \( W \) the exponential tension spline base functions (Schweikert, 1994) which are also called "Smith-Wilson" functions (Smith & Wilson, 2001):

\[ W(t, u) = \alpha \min(t, u) - \frac{1}{2} e^{-\alpha |t-u|} + \frac{1}{2} e^{-\alpha(t+u)} \]
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Lemma

For given \( u > 0 \) the function \( W(t, u) \) is the only twice continuously differentiable solution to

\[ \partial_t^2 W(t, u) = \alpha^2 W(t, u) - \alpha^3 \min(t, u) \quad (1) \]

which is zero for \( t = 0 \) and has a finite limit for \( t \to \infty \).
Covariance structure and Existence of Inverse

Functions $W$ are, according to EIOPA, related to covariance function of integrated Ornstein-Uhlenbeck processes but they do not match exactly (Andersson & Lindholm, 2013). In fact

**Proposition**

Let $Z_t$ be a standard Brownian Motion and $L_t$ be an Ornstein-Uhlenbeck process $L_t = \int_0^t e^{-\alpha(t-s)}dV_s$ with $V$ a Brownian Motion independent of $Z$ and $\alpha > 0$ a given constant. Then

$$W(t, u) = \alpha \text{cov}(Z_t + L_t, Z_u - L_u).$$
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This characterizes $W$ as a covariance between two different processes.

Also equals (auto)covariance process of a single Gaussian process on any finite interval $[0, T]$. 
Proposition

For a given $T > 0$ and $\alpha > 0$ let $(z_n)_{n \in \mathbb{N}}$ be the countably infinite number of solutions to the equation

$$z^3 \tan(z\alpha T) + (1 + z^2)^{\frac{3}{2}} \tanh((1 + z^2)^{\frac{3}{2}} \alpha T) = -1$$

and define

$$\psi_n(t) = z_n \frac{\sin(\alpha tz_n)}{\cos(\alpha Tz_n)} + \sqrt{1 + z_n^2} \frac{\sinh(\alpha t \sqrt{1 + z_n^2})}{\cosh(\alpha T \sqrt{1 + z_n^2})}$$

$$X_t = \frac{1}{\sqrt{\alpha}} \sum_{n=0}^{\infty} \frac{\psi_n(t) \epsilon_n}{\|\psi_n\|_2 z_n \sqrt{1 + z_n^2}}$$

with $(\epsilon_n)_{n \in \mathbb{N}}$ iid standard Gaussian.

Then $X$ has covariance function $W$ on domain $[0, T]$ i.e. $\mathbb{E}(X_t X_u) = W(t, u)$. 
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Corollary: matrix with elements $w_{ij} = W(u_i, u_j)$ is invertible.
Constrained variational problem

- Idea Smith & Wilson: interpolating discount curves

\[ \bar{\rho}(0, t) = (1 + g(t)) e^{-f_\infty t}, \quad g(t) = \sum_{j \in J} \eta_j \mathcal{W}(t, u_j) \]

should be required to be sufficiently smooth.
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- Functions \( W \) are solutions to variational problem

\[
\begin{align*}
\min_{g \in F_0} & \quad \mathcal{L}[g], \quad \mathcal{L}[g] := \int_0^\infty \left[ g''(s)^2 + \alpha^2 g'(s)^2 \right] ds \\
\end{align*}
\]

on the "Sobolev" space

\[
\begin{align*}
F_a & = \{ g \in C^2(\mathbb{R}^+) : g(0) = a, \ g' \in \mathcal{E}, \ g'' \in \mathcal{E} \} \\
\mathcal{E} & = \{ g \in L^2(\mathbb{R}^+) : \lim_{t \to \infty} g(t) = 0 \}.
\end{align*}
\]

and \( \eta = \tilde{C}^T(\tilde{C}W\tilde{C}^T)^{-1}m \), with \( \tilde{C}_{ij} = c_{ij}e^{-f_{\infty}u_j} \) and \( \tilde{m}_i = m_i - \sum_{j \in J} \tilde{C}_{ij}. \)
Unconstrained variational problem

- Our proposal: remove asymptotic constraint on forward rate but keep the optimization criterion as before. This "picks" the ultimate forward rate which creates "minimal tension" in discount curves.
Unconstrained variational problem

- Our proposal: remove asymptotic constraint on forward rate but keep the optimization criterion as before. This "picks" the ultimate forward rate which creates "minimal tension" in discount curves.
- We thus solve

$$
\min_{f_\infty} \min_{g \in \mathcal{H}_{f_\infty}} \mathcal{L}[g]
$$

on space

$$
\mathcal{H}_{f_\infty} = \left\{ g \in \mathcal{C}^2(\mathbb{R}_+) \mid g''(0) = 0, \quad \sum_{j \in \mathcal{J}} \tilde{C}_{ij} g(u_j) = \bar{m}_i, \quad \text{for all } i \in \mathcal{I} \right\}
$$
Unconstrained variational problem

Theorem

The optimized ultimate forward rate \( f = f_\infty \) solves

\[
(m - CD^f e)^T (CD^f WD^f C^T)^{-1} CD^f U \left( e + WD^f C^T (CD^f WD^f C^T)^{-1} (m - CD^f e) \right) = 0
\]

with \( W_{ij} = W(u_i, u_j) \), \( D_{ij}^f = e^{-f u_j} 1_{\{i=j\}} \), \( U_{ij} = u_j 1_{\{i=j\}} \), \( e_i = 1 \).

If the cashflow matrix \( C \) is invertible this simplifies to

\[
\sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{J}} (u_j \pi_j e^f u_j) W_{jk}^{-1} (\pi_k e^f u_k - 1) = 0
\]

with \( \pi = C^{-1} m \).

\[
A = @(D) \left( C*D*W*D*C' \right);
goal = @(D) \left( (m-C*D*e)' \right) * (A(D) \setminus (C*D*diag(u) * (e+(W*D*C')*(A(D) \setminus (m-C*D*e)))));
ufr = fzero(@(f) goal(diag(exp(-f*u))) , 0.02),
\]
Example: Euro Swap Rates, 2 Jan 2001
Example: Dutch regulator curve, 31 March 2013
Alternative formulation on different function space

- Last example shows that smoothest convergence discount curve does not translate into smoothest convergence forward rate. Given UFR philosophy that would be more natural criterion.
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- Our alternative formulation is therefore in terms of forward rates $g$

$$\min_{g \in \mathcal{H}} \mathcal{L}[g]$$

on function space

$$\mathcal{H} = \left\{ g \in C^2(\mathbb{R}_+) \mid g(0) = a, \quad g''(0) = \lim_{t \to \infty} g''(t) = 0, \quad \sum_{j \in J} c_{ij} e^{-\int_0^u g(s) \, ds} = m_i, \quad \text{for } i = 1, \ldots, J \right\}.$$
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\sum_{j \in J} c_{ij} e^{-\int_0^{u_j} g(s) \, ds} = m_i, \quad \text{for } i = 1, \ldots, J \end{array} \right\}.$$ 

Notice that we assume that short rate $g(0)$ is observed. It can be estimated during the optimization as well.
Solution

Theorem

A solution of this problem must take the form

\[ g(t) = g(0) + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \zeta_i c_{ij} \pi_j H(t, u_j), \]

\[ H(t, u) = 1 - e^{-\alpha t} \frac{\cosh(\alpha u) - 1}{\frac{1}{2} \alpha^2 u^2} + \mathbf{1}_{\{t \leq u\}} \left( \frac{\cosh(\alpha(u-t)) - 1 - \frac{1}{2} \alpha^2 (u-t)^2}{\frac{1}{2} \alpha^2 u^2} \right) \]

with the \((\zeta_i)_{i \in \mathcal{I}}\) and \((\pi_j)_{j \in \mathcal{J}}\) solving the equations

\[ m_i = \sum_{j \in \mathcal{J}} c_{ij} \pi_j, \quad -\ln \pi_k = g(0) u_k + \sum_{i \in \mathcal{I}} \zeta_i \sum_{j \in \mathcal{J}} \pi_j c_{ij} \int_0^{u_k} H(s, u_j) \, ds \]

Functions \(H\) start at \(H(0, u) = 0\) and converge to \(\lim_{t \to \infty} H(t, u) = 1\) with \(\partial^2_{1} H(0, u) = 0\). They are smoother than \(W\).
Example: Dutch regulator curve, 31 March 2013
Solution

The UFR follows directly from the optimization. Denote by

\[ y(u_k) = -\ln p(0, u_k)/u_k \]

the yield for maturity \( u_k \), and let \( y(u_0) := y(0) \) be the short rate.

**Theorem**

*If the cashflow matrix is invertible then*

\[ f_\infty = \sum_{k=0}^{n} v_k y(u_k) \]

*The coefficients \((v_k)\) equal*

\[ v_k = \sum_{j=1}^{n} G_{jk}^{-1}, \quad v_0 = 1 - \sum_{k=1}^{n} v_k, \quad G_{kj} = \frac{1}{u_k} \int_{0}^{u_k} H(s, u_j)ds. \]
Example: Euro Swap Rates 2001-2007

[Graphs showing Euro Swap Rates 2001-2007 with lines labeled UFR 4.2% and UFR opt disc curve.]

