PRICING LIFE INSURANCE CONTRACTS WITH EARLY EXERCISE FEATURES

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Options in life insurance

Options embedded in life insurance contracts:

- European style with possibly random maturity ⇒ Titanic option [Milevsky and Posner - JRI(01)] (minimum guarantees, bonus options, conversion options, . . . );
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If mortality risk can be diversified away (by pooling), then a Titanic option can be reduced to a portfolio of European options with different maturities; this does not apply to American options ⇒ valuation problem.
Approaches to valuation

- **Binomial Trees (and Extensions):** e.g. [Grosen and Jorgensen - IME(00)], [Bacinello - JRI(03), NAAJ(03), IME(05)];
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Complexity of the problem involved ⇒ oversimplified assumptions:

- no (or not realistic) mortality risk modelling;
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Monte Carlo simulation combined with LS regression (**LSM**, [Carrière - IME(96)], [Longstaff and Schwarz - RFS(01)]) allows to overcome such drawbacks.
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- numerical example:
  - unit-linked endowment insurance with **terminal** or **cliquet** guarantees;
  - interest rates: CIR reference portfolio: GBM+SV+J mortality: time dependent coefficients square root+J.
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- Value of the contract with surrender option (for fixed $\theta$): $V^r_0(\theta)$; the value of the contract is given by the optimal stopping problem

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where $T = \text{set of } \mathcal{G}\text{-stopping times}$.

- one can replace $\mathcal{G}$-s.t. with $\mathcal{F}$-s.t. or s.t. bounded by $\tau$. 
**LSM algorithm**

**Unbundling** of the contract:

\[ V_{0r}^* = V_0 + W_0^* \]

where \( V_0 \) = value of the contract without surrender and \( W_0^* \) = value of the surrender option (right to receive \( B^r \) and give up \( V \)).
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In order to compute \( V_{0}^{*} \) with backward dynamic programming, the LSM requires

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Convergence of the whole scheme is guaranteed if state variables are Markov, see [Clément et al. - FS(02)].
\[ V_0^r (\theta) = E^Q \left[ \int_0^\theta \frac{d(D_u + D_u^r(\theta))}{S_u^0} \right], \]

where \( S^0 \) is the money market account.
Method I: apply the algorithm directly to evaluate

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**Method II**: exploit the Cox setting \( \Rightarrow \) replace indicators with probabilities, i.e. discount sums at risk-adjusted rate \( r + \mu \):
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where \( d\hat{D}_u = dB_u^s + B_u^d \mu_u du \) and \( \hat{D}^r = B^r \) and \( \hat{S}^0 \) is the adjusted money-market account ⇒ contract without mortality.
Numerical examples

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$$B_t^s = F_T^g 1_{t \geq T} \quad B_t^d = F_t^g 1_{t < T} \quad B_t^r = F_t^h 1_{t < T},$$

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- **Cliquet guarantee:**

  \[
  F_t = F^l_t = S_0 \prod_{u=1}^{\lfloor t \rfloor} \max \left\{ \eta \left( \frac{S_u}{S_{u-1}} - 1 \right) + 1, e^g \right\}, \quad l = g, h
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- Interest rates (CIR): \( dr_t = k_r (\theta_r - r_t)dt + \sigma_r \sqrt{r_t} dW_t^r. \)
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\[
\begin{align*}
    dY_t &= \left( r_t - \frac{1}{2} Z_t - \lambda_J \mu_J \right) dt + \sqrt{Z_t} \left( \rho_{SZ} dW^Z_t + \rho_{Sr} dW^r_t \\
    & \quad + \sqrt{1 - \rho_{SZ}^2 - \rho_{Sr}^2} dW^S_t \right) + dJ_t \\
    dZ_t &= k_Z (\theta_Z - Z_t) dt + \sigma_Z \sqrt{Z_t} dW^Z_t
\end{align*}
\]

where \( W = (W^r, W^Z, W^S) \) is a standard B.m. in \( \mathbb{R}^3 \) independent of the compound Poisson \( J \) (arrival intensity \( \lambda_J \), lognormal(\( \mu_J, \sigma_J \)) jumps).
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- **Stochastic mortality:** left continuous version of

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\begin{align*}
  d\mu_t & = k_\mu (m(t) - \mu_t) dt + \sigma_\mu \sqrt{\mu_t} dW^\mu_t + dK_t
\end{align*}
\]

where \( m \) is a deterministic force of mortality, \( W^\mu \) is a standard B.m. independent of the compound Poisson \( K \) (arrival intensity \( \lambda_K \) and \( \exp(\gamma_K) \) jumps).
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- $T = 15$, $x = 40$;

- N. of simulations = 10000; forward discretization step = 1500, backward discretization step = 30.
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- Financial model:
  - $r_0 = 0.05, \ k_r = 0.6, \ \theta_r = 0.05, \ \sigma_r = 0.03$;
  - $Z_0 = 0.04, \ k_Z = 1.5, \ \theta_Z = 0.04, \ \sigma_Z = 0.4$;
  - $S_0 = 100, \ \rho_{ZS} = -0.7, \ \rho_{rS} = 0, \ \lambda_J = 0.5, \ \mu_J = 0, \ \sigma_J = 0.07$. 
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  - $S_0 = 100$, $\rho_{ZS} = -0.7$, $\rho_{rS} = 0$, $\lambda_J = 0.5$, $\mu_J = 0$, $\sigma_J = 0.07$.

- Demographic model: $m$ Weibull fitted against a SIM2001;
  $k_{\mu} = 0.5$, $\sigma_{\mu} = 0.03$, $\lambda_{\mu} = 0.1$, $\gamma_{\mu} = 100$.

- State variables: $\mu, r, S, Z$ (+ $F$ for the cliquet guarantee). Basis functions: polynomials in 4 (5) variables of grade 4.
Table 1: Surrender option premiums $W_0^*$ for different values of the minimum interest rate terminal guarantee at death or maturity ($g$) and at surrender ($h$).

<table>
<thead>
<tr>
<th>$h$</th>
<th>$V_0$</th>
<th>$g$ 0.00</th>
<th>0.01</th>
<th>0.02</th>
<th>0.03</th>
<th>0.04</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>107.07</td>
<td>8.25</td>
<td>6.37</td>
<td>4.16</td>
<td>2.10</td>
<td>0.52</td>
</tr>
<tr>
<td>0.01</td>
<td>109.41</td>
<td>8.02</td>
<td>5.09</td>
<td>2.70</td>
<td>0.68</td>
<td></td>
</tr>
<tr>
<td>0.02</td>
<td></td>
<td>6.93</td>
<td>3.49</td>
<td>0.99</td>
<td></td>
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<tr>
<td>0.03</td>
<td></td>
<td></td>
<td>5.23</td>
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<td>0.04</td>
<td></td>
<td></td>
<td></td>
<td>2.88</td>
<td></td>
<td></td>
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</tbody>
</table>
Table 2: Surrender option premiums $W_0^* (V_0)$ for different values of the cliquet guarantees ($g$) and ($\eta$).

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>$g = 0.00$</th>
<th>0.01</th>
<th>0.02</th>
<th>0.03</th>
<th>0.04</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>32.23</td>
<td>28.25</td>
<td>22.87</td>
<td>15.99</td>
<td>7.20</td>
</tr>
<tr>
<td></td>
<td>(65.82)</td>
<td>(69.86)</td>
<td>(75.37)</td>
<td>(82.73)</td>
<td>(92.29)</td>
</tr>
<tr>
<td>0.4</td>
<td>9.72</td>
<td>4.88</td>
<td>0.77</td>
<td>0.11</td>
<td>0.01</td>
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<tr>
<td></td>
<td>(90.25)</td>
<td>(95.35)</td>
<td>(101.51)</td>
<td>(108.99)</td>
<td>(118.05)</td>
</tr>
<tr>
<td>0.6</td>
<td>0.04</td>
<td>0.01</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>(122.87)</td>
<td>(129.54)</td>
<td>(137.26)</td>
<td>(146.22)</td>
<td>(156.67)</td>
</tr>
<tr>
<td>0.8</td>
<td>0.01</td>
<td>0.01</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>(166.13)</td>
<td>(174.89)</td>
<td>(184.80)</td>
<td>(196.05)</td>
<td>(208.86)</td>
</tr>
<tr>
<td>1.0</td>
<td>0.00</td>
<td>0.01</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>(223.14)</td>
<td>(234.63)</td>
<td>(247.45)</td>
<td>(261.79)</td>
<td>(277.87)</td>
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</table>